12/3 Recitation Solutions

1. Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$.

   (a) Show that $\mathcal{B}$ is a basis of $\mathbb{R}^3$.

   Let $M_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$. Using Julia, we see that $\det M_{\mathcal{B}} = -1$; hence $M_{\mathcal{B}}$ is invertible. It follows that the vectors are linearly independent and span $\mathbb{R}^3$.

   (b) Find the $\mathcal{B}-$coordinate vector of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

   Using Julia, we find $M_{\mathcal{B}}^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$. Hence

$$[\mathbf{v}]_{\mathcal{B}} = M_{\mathcal{B}}^{-1}[\mathbf{v}]_{\mathcal{E}} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

2. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x + 4y \\ x + y \end{bmatrix}$ (expressed in the standard basis).

   (a) Write down the standard matrix A for T (written in the standard basis)

   $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ since $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 4y \\ x + y \end{bmatrix} = T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$.

   (b) Write down the associated matrix D of T written in the basis $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right\}$.

   The change of basis matrix from $\mathcal{B}$ to $\mathcal{E}$ is $M_{\mathcal{B}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}$. Hence the associated matrix is

$$D = M_{\mathcal{B}}^{-1} A M_{\mathcal{B}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 3 \end{bmatrix}$$

(c) Diagonalize A without finding eigenvalues or eigenvectors.

Notice that D from part (b) is diagonal. Multiplying $D = M_{\mathcal{B}}^{-1} A M_{\mathcal{B}}$ on the left by $M_{\mathcal{B}}$ and on the right by $M_{\mathcal{B}}^{-1}$ yields $A = M_{\mathcal{B}} D M_{\mathcal{B}}^{-1}$. Since D is diagonal, this is a diagonalization of A.

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation with associated matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix}$ (written in the standard basis). Find a basis for which the associated matrix is diagonal.

We can diagonalize A to find such a matrix. Since A is upper triangular, its eigenvalues are 1, 2, and -1. One can find that

$$E_1(A) = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}, \quad E_2(A) = \text{Span}\left\{ \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix} \right\}, \quad E_{-1}(A) = \text{Span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ (check).}$$

Let $P = \begin{bmatrix} 1 & -7 & 2 \\ 0 & -2 & 1 \\ 0 & 6 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. Then $A = PDP^{-1}$ or $D = P^{-1}AP$.

Let $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \right\}$. Then $M_{\mathcal{B}} = P$ and so $D = M_{\mathcal{B}}^{-1} A M_{\mathcal{B}}$; hence $\mathcal{B}$ is a basis for which the associated matrix is diagonal.

4. In class we saw that if $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, then every vector in $\mathbb{R}^n$ can be written *uniquely* as a linear combination of the basis vectors; the key was linear independence. Show that if $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ spans $\mathbb{R}^n$, but is not linearly independent, then every vector in $\mathbb{R}^n$ can be written as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_k$ in infinitely many ways.

Since $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ is not linearly independent, the matrix $A = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_k \end{bmatrix}$ does not have a pivot in every column. However, since the vectors span $\mathbb{R}^n$, A has a pivot in every row. Hence A is an $n \times k$ matrix with $k > n$. Recall that for any $\mathbf{x} \in \mathbb{R}^k$, $A\mathbf{x}$ is a linear combination of the columns of A (i.e. $\mathbf{b}_1, \ldots, \mathbf{b}_k$). Hence for any $\mathbf{c} \in \mathbb{R}^n$, the number of linear combinations of $\mathbf{b}_1, \ldots, \mathbf{b}_k$ that equal $\mathbf{c}$ is the number of solutions to $A\mathbf{x} = \mathbf{c}$. Since A has a pivot in every row, this system has a solution. Since A does not have a pivot in every column, there is a free variable and so there are infinitely many solutions. Hence $\mathbf{c}$ can be expressed as a linear combination of $\mathbf{b}_1, \ldots, \mathbf{b}_k$ in infinitely many ways.